MATHEMATICS

THE WEIGHTED DIEUDONNÉ THEOREM FOR DENSITY IN TENSOR PRODUCTS [1])

BY

JOÃO B. PROLLA

(Communicated by Prof. A. C. ZAANEN at the meeting of November 28, 1970)

1. *Introduction*

The objective of this paper is twofold. Firstly, to present an alternate proof of the weighted Dieudonné theorem for density in tensor products due to NACHBIN (Theorem 1, § 23, [4]), and secondly, to extend it to weighted spaces of vector-valued functions.

The classical Dieudonné theorem for density in tensor products can be proved directly using partitions of unity, or alternatively using the Stone-Weierstrass theorem. (For both proofs, see § 20, [4]). The direct proof of the weighted Dieudonné theorem due to Nachbin is quite long, the main idea still being partitions of the unity, now by means of functions belonging to one of the weighted spaces. On the other hand, the Stone-Weierstrass theorem is subsumed by the bounded case of the weighted approximation problem. So it seems natural to ask whether an alternate proof of the weighted Dieudonné theorem can be found based on the bounded case of the weighted approximation problem. In § 2 we show how to accomplish this.

The most natural setting for the vector-valued case is the one in which the range spaces are locally convex spaces which are topological modules over some topological algebra A. One then considers the A-tensor product of these modules. The case in which the range spaces are only vector spaces over the same field $\mathbf{K}$ of scalars is then a particular case, namely $A=\mathbf{K}$. Another interesting case occurs when one of the range spaces is the algebra A itself and the other range space is an essential A-module, i.e. every element in the module can be approximated by finite sums of products ax, where a belongs to the algebra and x to the module.

Throughout this paper we use the notations of [4] and [5], where all symbols and terms used here without explanation or definition are explained or defined.

2. Let E and F be two completely regular Hausdorff spaces and V

[1]) This research was supported by the National Science Foundation, Grant No. GP22713.

and W two directed sets of upper semi-continuous positive real-valued functions on E and F respectively. Let $V \times W$ denote the set of all functions $v \otimes w$ on $E \times F$, where $v \in V$ and $w \in W$. Recall that $v \otimes w$ is the function defined on $E \times F$ by $(x, y) \to v(x)w(y)$. Let $CV_\infty(E) \otimes CW_\infty(F)$ denote the vector space of all finite sums of functions of the form $f \otimes g$, where $f \in CV_\infty(E)$ and $g \in CW_\infty(F)$.

2.1. Theorem (Nachbin). *$CV_\infty(E) \otimes CW_\infty(F)$ is dense in*

$$C(V \times W)_\infty(E \times F).$$

For our proof we need the following.

2.2. Lemma. *Let A be a separating and self-adjoint subalgebra of $C_b(X)$, where X is a completely regular Hausdorff space. Let L be a vector subspace of $CV_\infty(X)$ which is an A-module, where V is a directed set of weights over X. A function $f \in CV_\infty(X)$ belongs to the closure of L if, and only if, for each $x \in X$ such that $f(x) \neq 0$ and $v(x) > 0$ for some $v \in V$, there exists $w \in L$ such that $w(x) \neq 0$.*

Proof. Since $A \subset C_b(X)$, the A-module L is localizable under A in $CV_\infty(X)$. (See Theorem 1, § 30, Nachbin [4].) Hence $f \in CV_\infty(X)$ belongs to the closure of L in $CV_\infty(X)$ if, and only if, $f(x)$ belongs to the closure of $L(x)$ in $CV_\infty(\{x\})$, for each $x \in X$. Now $f(x)$ belongs to the closure of $L(x)$ whenever $f(x) = 0$ or $v(x) = 0$ for all $v \in V$. Indeed, in the second case $CV_\infty(\{x\})$ is equipped with the 0-seminorm only.

Proof of Theorem 2.1. Let A denote the subalgebra $C_b(E) \otimes$ $\otimes C_b(F)$ of $C_b(E \times F)$. It is clear that A is self-adjoint and separating on $E \times F$. For every pair $f \in CV_\infty(E)$ and $g \in CW_\infty(F)$, $f \otimes g$ belongs to $C(V \times W)_\infty(E \times F)$. (See Proposition 1, § 23, Nachbin [4].) Let L denote $CV_\infty(E) \otimes CW_\infty(F)$. The fact that $CV_\infty(E)$ is a $C_b(E)$-module and $CW_\infty(F)$ is a $C_b(F)$-module implies that L is an A-module. Let $f \in C(V \times W)_\infty(E \times F)$ and $(x, y) \in E \times F$ be such that $f(x, y) \neq 0$ and $(v \otimes w)(x, y) > 0$ for some $v \otimes w \in V \times W$. It follows that $v(x) > 0$ and $w(y) > 0$. By Lemma 1, § 23, Nachbin [4], the partial functions $t \to w_1(t) = f(t, y)$ and $u \to w_2(u) = f(x, u)$ belong to $CV_\infty(E)$ and $CW_\infty(F)$ respectively. Hence $w = w_1 \otimes w_2$ belongs to L and $w(x, y) = [f(x, y)]^2 \neq 0$. By the above Lemma, f belongs to the closure of L. Thus L is dense in $C(V \times W)_\infty(E \times F)$.

3. To extend the above result to *vector-valued functions* we need the analogue of Lemma 2.2, namely the following result.

3.1. Lemma. *Let X, V and A as in* Lemma 2.2 *and B a locally convex Hausdorff space. Let $L \subset CV_\infty(X, B)$ be a vector subspace which is an A-module. A function $f \in CV_\infty(X, B)$ belongs to the closure of L if, and only if, for every $x \in X$ such that $v(x) > 0$ for some $v \in V$, the vector $f(x)$ is in the closure of $L(x)$ in B.*

Proof. Since $A \subset C_b(X)$, the A-module L is localizable under A in $CV_\infty(X, B)$. (See Theorem 10, § 6, NACHBIN, MACHADO, PROLLA [5].) Hence $f \in CV_\infty(X, B)$ belongs to the closure of L if, and only if, for every $x \in X$ the vector $f(x)$ belongs to the closure of $L(x)$ in $CV_\infty(\{x\}, E)$. If $f \in CV_\infty(X, B)$ belongs to the closure of L and $x \in X$ is such that $v(x) > 0$ for some $v \in V$, let $\varepsilon > 0$ and p be a continuous seminorm on B. There exists $w \in L$ such that $\sup \{v(t)p(f(t)-w(t)); t \in X\} < \varepsilon v(x)$. In particular, $v(x)p(f(x)-w(x)) < \varepsilon v(x)$, i.e. $p(f(x)-w(x)) < \varepsilon$ and $f(x)$ belongs to the closure of $L(x)$ in B. Conversely, suppose $f(x)$ belongs to the closure of $L(x)$ in B for every $x \in X$ such that $v(x) > 0$ for some $v \in V$. Let $t \in X$ and $u \in V$, $\varepsilon > 0$ and p a continuous seminorm on B be given. If $u(t) = 0$, then $u(t)p(f(t)-w(t)) < \varepsilon$ for any $w \in L$. If $u(t) > 0$, by the above hypothesis there is $w \in L$ such that $p(f(t)-w(t)) < \varepsilon/u(t)$. Hence $u(t)p(f(t)-u(t)) < \varepsilon$. In any case $f(t)$ belongs to the closure of $L(t)$ in $CV_\infty(\{t\}, B)$ for every $t \in X$, and by the remark made at the beginning of the proof, f belongs to the closure of L in $CV_\infty(X, B)$, q.e.d.

3.2. Corollary. *$CV_\infty(X) \otimes B$ is dense in $CV_\infty(X, B)$.*

Proof. Let $A = C_b(X)$ and $L = CV_\infty(X) \otimes B$. Since $CV_\infty(X)$ is a $C_b(X)$-module, it follows that L is an A-module. Let $f \in CV_\infty(X, B)$ and $x \in X$ be such that $v(x) > 0$ for some $v \in V$. If $f(x) = 0$, then $f(x) \in L(x)$ obviously. If $f(x) \neq 0$, let φ be a continuous linear functional on B such that $\varphi(f(x)) \neq 0$. We claim that $\varphi \circ f$ belongs to $CV_\infty(X)$. Let $u \in V$ and $\varepsilon > 0$ be given. Since $f \in CV_\infty(X, B)$ there exists $K \subset X$ compact such that $u(t)|\varphi(f(t))| < \varepsilon$ whenever $t \in X$ is outside of K. Hence $\varphi \circ f$ belongs to $CV_\infty(X)$ as claimed. Let $g \in CV_\infty(X)$ be the function $\varphi \circ f/\varphi(f(x))$. Then $g(x) = 1$ and $g \otimes f(x) \in CV_\infty(X) \otimes B$ is such that $[g \otimes f(x)](x) = f(x)$, i.e. $f(x) \in L(x)$. By Lemma 3.1, $CV_\infty(X) \otimes B$ is dense in $CV_\infty(X, B)$.

3.3. Corollary. *Let E, F, V and W be as in* Theorem 2.1. *Then $CV_\infty(E) \otimes CW_\infty(F, B)$ is dense in $C(V \times W)_\infty(E \times F, B)$.*

Proof. Immediate from Corollary 3.2 combined with Theorem 2.1.

4. In this section we extend Theorem 2.1 to the case of vector-valued functions, when the range spaces are *locally convex A-modules.*

4.1. Definition. *Let A be a topological algebra. A topological module over A, also said a topological A-module, is a topological vector space B which is a (left or right) module over A in the usual algebraic sense, such that the bilinear map $A \times B \to B$ (whose value at (a, b) we will denote by ab) is continuous.*

If both A and B are locally convex spaces, the bilinear map $A \times B \to B$ is continuous if, and only if, given any continuous seminorm p on B, there exist a continuous seminorm q_1 on A, a continuous seminorm p_2 on B and a constant $k > 0$ such that $p(ab) \leqslant kq_1(a)p_2(b)$ for all $a \in A$, $b \in B$.

4.2. Examples. (a) Every topological vector space is a topological module over the scalar field.

(b) Every topological algebra with jointly continuous multiplication is a topological module over itself.

(c) If A is a Banach algebra and B is a Banach A-module in the sense of HEWITT [3] or RIEFFEL [7], then B is a topological A-module.

4.3. Definition. *A net $\{x_i\}$ of elements of a topological algebra A such that $x_i a \to a$ for every $a \in A$, is called an approximate left unit. Similarly one defines approximate right and two-sided units.*

An approximate left (or right) unit is said to be *bounded* if the net $\{x_i\}$ is bounded. If A is locally convex, $\{x_i\}$ is bounded if, and only if, $\sup \{q(x_i); i \in I\} < \infty$ for each continuous seminorm q on A. If moreover there exists a constant $K < +\infty$ such that $\sup \{q(x_i); i \in I\} \leqslant K$ for all continuous seminorms q which belong to a family of seminorms which determines the topology of A, then $\{x_i\}$ is said to be *uniformly bounded.*

4.4. Remark. Let R be any ring and let M be a left R-module. If R has a unit element e, and if $em = m$ for all $m \in M$, then M is said to be a *unital module.* This motivates the following.

4.5. Definition. *Let A be a topological algebra with an approximate left unit $\{a_i\}$ and let B be a (left or right) topological module over A. If $a_i b \to b$ for all $b \in B$, then B is said to be an approximate left-unital module.* Similarly one defines *approximate right-unital modules.* If $\{a_i\}$ is an approximate two-sided unit and $a_i b \to b$ for all $b \in B$, then B is said to be an *approximate unital module.*

4.6. Definition. *Let A be a topological algebra and let B be a (left or right) topological module over A. We say that B is an essential A-module if the vector space spanned by $AB = \{ab; a \in A, b \in B\}$ is dense in B.*

4.7. Theorem. *Let A be a locally convex topological algebra with a bounded approximate left unit $\{x_i\}$, and let B be a locally convex space which is a (left or right) topological A-module. Then the following are equivalent:*

(a) *B is an essential A-module.*

(b) *B is an approximate left-unital module.*

Proof. It is evident that (b) implies (a). Let $b_0 \in B$. Given $\varepsilon > 0$ and p a continuous seminorm on B, there exist a continuous seminorm q on A, a continuous seminorm p' on B and a constant $k > 0$ such that $p(ab) \leqslant kq(a)p'(b)$ for all $a \in A$, $b \in B$. Since $\{x_i\}$ is bounded, there exists a constant $c > 0$ such that $q(x_i) \leqslant c$ for all $i \in I$. Since B is an essential A-module, there exists $a_1 \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$ such that

$$p(b_0 - \sum a_k b_k) < \varepsilon/3$$

and

$$p'(b_0 - \sum a_k b_k) < \varepsilon/3ck.$$

Choose now $j_0 \in I$ such that $i > j_0$ implies

$$q(a_k - x_i a_k) < \varepsilon/(3np'(b_k))$$

for all $1 \leqslant k \leqslant n$. Then for $i > j_0$ we have

$$\begin{aligned} p(b_0 - x_i b_0) \leqslant p(b_0 - \sum a_k b_k) + p(\sum a_k b_k - x_i(\sum a_k b_k)) \\ + p(x_i(\sum a_k b_k) - x_i b_0) < \\ < \varepsilon/3 + \sum p((a_k - x_i a_k) b_k) + q(x_i) p'(\sum a_k b_k - b_0) k \\ < \varepsilon/3 + \sum q(a_k - x_i a_k) p'(b_k) + c \cdot \varepsilon/3c \\ < \varepsilon/3 + n \cdot \varepsilon/3n + \varepsilon/3 = \varepsilon. \end{aligned}$$

This ends the proof that (a) $\Rightarrow$ (b).

4.8. Remarks. For many topological modules the properties (a) and (b) of Theorem 4.7 are equivalent to the stronger property

$$\text{(c)} \quad AB = \{ab;\, a \in A,\ b \in B\} = B.$$

When (c) is valid we say that the A-module B has the *factorization property*. Clearly any unital module has the factorization property. Cohen [1] proved that every Banach algebra with a bounded approximate unit has the factorization property, a result that was extended by Hewitt [3] to bounded approximate left-unital Banach modules. The notion of essential A-modules was introduced by Rieffel [7]. Theorem 4.7 above is a straightforward generalization of the corresponding result of Rieffel for Banach modules. The equivalence (c) and (a) or (b) has been established for Fréchet algebras (Craw [2]) and Fréchet modules (Ovaert [6], Summers [8]).

4.9. Definition. Let A be a locally convex topological algebra, and let M and N be two locally convex spaces which are topological modules over A. Then $M \otimes_A N$ is defined to be the quotient locally convex space $(M \otimes N)/D$, where $M \otimes N$ is the tensor product of the vector spaces M and N endowed with the projective tensor product topology, and D is the closed linear subspace of $M \otimes N$ spanned by elements of the form $(ax \otimes y - x \otimes ay)$, $a \in A$, $x \in M$, $y \in N$.

Now let E, F, V and W be as in § 2, and let A, M and N be as in the above definition. If $f: E \to M$ and $g: F \to N$, then $f \otimes_A g$ denotes the map $(x, y) \to f(x) \otimes_A g(y)$ from $E \times F$ into $M \otimes_A N$. If f and g are continuous, $f \otimes_A g$ is also continuous. Moreover, if $f \in CV_\infty(E, M)$ and $g \in CW_\infty(F, N)$, then $f \otimes_A g$ belongs to the space $C(V \times W)_\infty(E \times F, M \otimes_A N)$, the generalization of Proposition 1, § 23, Nachbin [4] being straightforward. We will denote by $CV_\infty(E, M) \otimes_A CW_\infty(F, N)$ the vector subspace of $C(V \times W)_\infty(E \times F, M \otimes_A N)$ consisting of all finite sums of mappings of the form $f \otimes_A g$, where $f \in CV_\infty(E, M)$, $g \in CW_\infty(F, N)$.

4.10. Theorem. *$CV_\infty(E, M) \otimes_A CW_\infty(F, N)$ is dense in $C(V \times W)_\infty$ $(E \times F, M \otimes_A N)$.*

Proof. The subspace $CV_\infty(E, M) \otimes_A CW_\infty(F, N)$ contains

$$[CV_\infty(E) \otimes CW_\infty(F)] \otimes (M \otimes_A N).$$

By Theorem 2.1, $CV_\infty(E) \otimes CW_\infty(F)$ is dense in $C(V \times W)_\infty(E \times F)$. Hence, $CV_\infty(E, M) \otimes_A CW_\infty(F, N)$ is dense in $C(V \times W)_\infty(E \times F) \otimes \otimes (M \otimes_A N)$, which is dense in $C(V \times W)_\infty(E \times F, M \otimes_A N)$, by Corollary 3.2.

4.11. Corollary. *Let E, F, V and W be as in* Theorem 2.1 *and let M and N be two locally convex spaces. Then $CV_\infty(E, M) \otimes CW_\infty(F, N)$ is dense in $C(V \times W)_\infty(E \times F, M \otimes N)$.*

Proof. Let $A = \mathbf{K}$, the scalar field of M and N.

4.12. Corollary. *Let A be a locally convex topological algebra and let M be a locally convex space which is an essential topological module over A. Then $CV_\infty(E, A) \otimes CW_\infty(F, M)$ is dense in $C(V \times W)_\infty(E \times F, M)$.*

Proof. Both A and M are $\mathbf{K}$-modules and $A \otimes_{\mathbf{K}} M$ is the linear span of AM in M, which is dense in M, since M is essential.

The author wishes to thank Dr. Leopoldo Nachbin for a suggestion which led to an improvement of this paper.

University of Rochester
Rochester, New York
14627

REFERENCES

1. Cohen, P. J., Factorization in group algebras, Duke Math. J. **26**, 199–205 (1959).
2. Craw, I. G., Factorization in Fréchet algebras, J. London Math. Soc. **44**, 607–611 (1969).
3. Hewitt, E., The ranges of certain convolution operators, Math. Scand. **15**, 147–155 (1964).
4. Nachbin, L., Elements of Approximation Theory, Van Nostrand, Princeton (1967).
5. ———, S. Machado and J. B. Prolla, Weighted approximation, vector fibrations and algebras of operators, J. de Math. Pures et Appl. (to appear).
6. Ovaert, J. L., Factorisation dans les algèbres et modules de convolution, C. R. Acad. Sci. Paris, **265** série **A**, 534–535 (1967).
7. Rieffel, M. A., Induced Banach representations of Banach algebras and locally compact groups, J. Functional Anal. **1**, 443–491 (1967).
8. Summers, W. H., Factorization in Fréchet spaces, Studia Math. (to appear).